SURGERY FORMULA FOR THE RENORMALIZED EULER CHARACTERISTIC OF HEEGAARD FLOER HOMOLOGY

RAIF RUSTAMOV

Abstract. We prove a surgery formula for renormalized Euler characteristic of Ozsváth and Szabó. The equality $\widehat{\chi} = SW$ between this Euler cahracteristic and the Seiberg-Witten invariant follows for rational homology three-spheres.

1. Introduction

In [13] and [11] topological invariants for closed oriented three manifolds and cobordisms between them were defined by using a construction from symplectic geometry. The resulting Floer homology package has many properties of a topological quantum field theory.

Another such Floer homology package comes from Seiberg-Witten theory [5], [7]. Similarity of properties of the Ozsváth-Szabó and Seiberg-Witten theories and also calculations heavily support the conjecture that these invariants are equivalent.

In this paper we will concentrate on a numerical invariant of rational homology spheres obtained from the Heegaard Floer homology package - the renormalized Euler characteristic, $\widehat{\chi}$. It is already known that for integral homology spheres $\widehat{\chi}$ is equal to Casson's invariant [14], which is also the case for the Seiberg-Witten invariant of integral homology spheres [6]. Calculations of [9] push this equivalence further to the Lens spaces and Seifert manifolds. Thus, it is tempting to establish this equiality in its whole generality. To this end we prove a surgery formula for $\widehat{\chi}$. This formula and several other properties of $\widehat{\chi}$ and the related invariant χ^{trunc} together fit into the framework of [10] to give equivalence between $\widehat{\chi}$ and the Reidemester-Turaev torsion normalized by the Casson-Walker invariant. This also implies the equality $\widehat{\chi} = SW$.

The organization of the paper is as follows: the required preliminaries are presented in Section 2. The surgery theorem is formulated and its applications are given in Section 3. The paper finishes with the proof of the surgery formula in Section 4.

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2. Preliminaries

Correction terms and Euler characteristics Let Y be a rational homology sphere, $\mathfrak{t}$ be a Spin^c structure on it. We can consider Heegaard Floer homology group $HF^+(Y,\mathfrak{t})$. This is a $\mathbb{Q}$ graded module over $Z[U]$. We can also consider a simpler version, $HF^\infty(Y,\mathfrak{t})$, for which one can prove

$$HF^\infty(Y,\mathfrak{t}) \cong Z[U,U^{-1}] \tag{1}$$

for each $\mathfrak{t}$ structure. There is a natural $Z[U]$ equivariant map

$$\pi\colon HF^\infty(Y,\mathfrak{t}) \longrightarrow HF^+(Y,\mathfrak{t})$$

which is zero in sufficiently negative degrees and an isomorphism in all sufficiently positive degrees. $HF^+_{\mathrm{red}}(Y,\mathfrak{t})$ is defined as

$$HF^+_{\mathrm{red}}(Y,\mathfrak{t}) = HF^+(Y,\mathfrak{t})/\mathrm{Im}\pi.$$

Let $d(Y,\mathfrak{t})$ be the *correction term* defined as the minimal degree of any non-torsion class of $HF^+(Y,\mathfrak{t})$ lying in the image of π. Main object of our study, the *renormalized Euler characteristic* $\widehat{\chi}(Y,\mathfrak{t})$ is defined by

$$\widehat{\chi}(Y,\mathfrak{t}) = \chi(HF^+_{\mathrm{red}}(Y,\mathfrak{t})) - \frac{1}{2}d(Y,\mathfrak{t}).$$

When Y is a rational homology $S^1 \times S^2$ there is a related numerical invariant χ^{trunc} as follows. Define $\chi^{\mathrm{trunc}}(Y,\mathfrak{t}) = \chi(HF^+(Y,\mathfrak{t}))$ for non-torsion $\mathfrak{t}$. If $\mathfrak{t}$ is torsion then let $d(Y,\mathfrak{t})$ be the minimal degree of any non-torsion class of $HF^+(Y,\mathfrak{t})$ coming from $HF^\infty_{\mathrm{ev}}(Y,\mathfrak{t})$. The structure of HF^∞ for homology $S^1\times S^2$ implies that $\chi(HF^+_{\leq d(Y,\mathfrak{t})+2N+1}(Y,\mathfrak{t}))$ is independent of N for sufficiently large N. We let $\chi^{\mathrm{trunc}}(Y,\mathfrak{t})$ denote the value of this Euler characteristic.

One can express χ^{trunc} in terms of Turaev torsion function [18]

$$\tau_Y\colon \mathrm{Spin}^c(Y) \longrightarrow \mathbb{Z}.$$

It is proved in [12] that for any $\mathfrak{t}$,

$$\chi^{\mathrm{trunc}}(Y,\mathfrak{t}) = -\tau(Y,\mathfrak{t}).$$

For the precise statement and the sign issues for Turaev function we refer to Proposition 10.14 of [12].

In what follows, λ denotes the *Casson-Walker* invariant normalized by $\lambda(\Sigma(2,3,5)) = -1$, where $\Sigma(2,3,5)$ is oriented as the boundary of the negative definite E_8 plumbing.

Surgery Here we set up our framework for surgeries. We directly follow [16]. Let X be an oriented three-manifold with a torus boundary and $H_1(X;\mathbb{R}) \cong \mathbb{R}$. The map $H_1(\partial X;\mathbb{Z}) \longrightarrow H_1(X;\mathbb{Z})$ has one-dimensional kernel. Let ℓ' denote a generator for the kernel, $d(X) > 0$ denote its divisibility, and let ℓ be the element ℓ'/d. We call ℓ the *longitude*.

Fix a homology class $m \in H_1(\partial X)$ with $m \cdot \ell = 1$. For a pair of relatively prime integers (p,q), the manifold $Y_{p/q}$ is obtained from X by attaching a $S^1 \times D$ with

$\partial D = pm + q\ell$, and let $Y = Y_{1/0}$. Note that in general $Y_{p/q}$ depends on a choice of m, but $Y_0 = Y_{0/1}$ does not. Note also that Y_0 is a rational homology $S^1 \times S^2$, while all the other $Y_{p/q}$ are rational homology spheres.

There is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Spin}^c(Y_0) \longrightarrow \mathrm{Spin}^c(X) \longrightarrow 0, \tag{2}$$

by which we mean that the subgroup $\mathbb{Z} \subset H^2(Y_0;\mathbb{Z})$ generated by the Poincaré dual to m (viewed as a subset of Y_0) acts freely on $\mathrm{Spin}^c(Y_0)$, and its quotient is naturally identified (under restriction to $X \subset Y_0$) with $\mathrm{Spin}^c(X)$.

Thus, each Spin^c structure $\mathfrak{s}$ on X has a natural level $y = y(\mathfrak{s}) \in \mathbb{Z}/d\mathbb{Z}$ defined as follows. Let $\mathfrak{b}$ be any Spin^c structure on Y_0 whose restriction is $\mathfrak{a}$, and consider its image in

$$\underline{\mathrm{Spin}^c}(Y_0)/\mathbb{Z}(\mathrm{PD}[m]) \cong \mathbb{Z}/d\mathbb{Z},$$

where $\underline{\mathrm{Spin}^c}(Y_0)$ is the group of Spin^c structures modulo the action of the torsion subgroup of $H^2(Y_0;\mathbb{Z})$.

Furthermore, for any of the $Y_{p/q}$, the map $\mathrm{Spin}^c(Y_{p/q})$ to $\mathrm{Spin}^c(X)$ is surjective, and its fibers consist of orbits by a cyclic group generated by the Poincaré dual to the knot which is the core of the complement $Y_{p/q} - X$ (for $Y = Y_{1/0}$, this fiber has order $d = d(X)$). For a fixed Spin^c structure $\mathfrak{a}$ on X, let $\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})$ denote the set of Spin^c structures $\mathfrak{b} \in \mathrm{Spin}^c(Y_{p/q})$ whose restriction to X is $\mathfrak{a}$.

3. Surgery formula and its applications

Our main theorem is the following surgery formula for the Euler characteristic.

Theorem 3.1. *For integers p, q, d, y with $p \neq 0$, p and q relatively prime, $d > 0$ and $0 \leq y < d$, there is quantity $\epsilon(p,q,d,y) \in \mathbb{Q}$ with the following property. Let X be an oriented rational homology $S^1 \times D$, with divisibility $d(X) = d$, and choose m, ℓ as described in the previous section. Fixing any* Spin^c *structure $\mathfrak{a}$ over X with level $y(\mathfrak{a}) = y$, we have the relation:*

$$\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})} \widehat{\chi}(Y_{p/q},\mathfrak{b}) = p\left(\sum_{\mathfrak{c}\in\mathrm{Spin}^c(Y;\mathfrak{a})} \widehat{\chi}(Y,\mathfrak{c})\right) - q\left(\sum_{\mathfrak{d}\in\mathrm{Spin}^c(Y_0;\mathfrak{a})} \chi^{\mathrm{trunc}}(Y_0,\mathfrak{d})\right) + \\ + \epsilon(p,q,d,y).$$

Corollary 3.2. *For X as above,*

$$\left(\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q})} \widehat{\chi}(Y_{p/q},\mathfrak{b})\right) = p\left(\sum_{\mathfrak{c}\in\mathrm{Spin}^c(Y)} \widehat{\chi}(Y,\mathfrak{c})\right) + q(\sum_{i=1}^{\infty} a_i i^2) \\ + |\mathrm{Tors} H_1(X;\mathbb{Z})|\epsilon(p,q,d),$$

where $d = d(X)$, a_i are the coefficients of the symmetrized Alexander polynomial of Y_0, normalized so that

$$A(1) = |\mathrm{Tors} H^2(Y_0;\mathbb{Z})|,$$

and

$$\epsilon(p,q,d) = \sum_{y=0}^{d-1} \frac{\epsilon(p,q,d,y)}{d}.$$

Proof. This follows from the surgery formula and the fact that $\chi^{\mathrm{trunc}}(Y_0,\mathfrak{t}) = -\tau(Y_0,\mathfrak{t})$. □

Theorem 3.3. *For any rational homology three-sphere M we have*

$$\sum_{\mathfrak{t}\in \mathrm{Spin}^c(M)} \widehat{\chi}(M,\mathfrak{t}) = \big|H_1(M;\mathbb{Z})\big|\lambda(M),$$

where $\lambda(M)$ is the Casson-Walker invariant of M.

Proof. We already have a surgery formula for $\sum_{\mathfrak{t}\in \mathrm{Spin}^c(Y)} \widehat{\chi}(Y,\mathfrak{t})$. The scaled Casson-Walker invariant

$$\lambda'(Y) = |H_1(Y;\mathbb{Z})|\lambda(Y)$$

satisfies a similar formula with possibly different constants $\epsilon'(p,q,d)$, see [16]. In fact, we have

$$\lambda'(Y_{p/q}) = p\lambda'(Y) + q\left(\sum_{j\geq 1} a_j j^2\right) + |\mathrm{Tors} H_1(X;\mathbb{Z})|\left(\frac{q(d^2-1)}{24d} - \frac{pd\cdot s(q,p)}{2}\right),$$

i.e. $\epsilon'(p,q,d) = \left(\frac{q(d^2-1)}{24d} - \frac{pd\cdot s(q,p)}{2}\right)$. Thus, it remains to show that

$$\epsilon(p,q,d) = \epsilon'(p,q,d).$$

For $d = 1$ we can use a model calculation on $Y = S^3$ with the surgery made on the unknot. Since $S^3_{p/q} = L(-p,q)$, by [9] (see also [17]) we have

$$\sum_{\mathfrak{t}\in \mathrm{Spin}^c(L(p,q))} d(L(-p,q),\mathfrak{t}) = p\cdot s(q,-p) = p\cdot s(q,p).$$

Taking into account that $HF^+_{\mathrm{red}}(L(-p,q)) \cong 0$ it follows that in this case

$$\sum_{\mathfrak{t}\in \mathrm{Spin}^c(S^3_{p/q})} \widehat{\chi}(S^3_{p/q},\mathfrak{t}) = -\frac{p\cdot s(p,q)}{2}.$$

Plugging this into the surgery formula we get

$$\epsilon(p,q,1) = -\frac{p\cdot s(p,q)}{2}$$

as needed.

To complete the proof, one shows that $\epsilon(p,q,d)$ is determined by the surgery formula and the values of $\epsilon(p,q,1)$. This is done by considering the Seifert manifold $M(n,1,-n,1,q,-p)$. It can be obtained from $M(n,1,-n,1,0,1)$ by (p,q,n) surgery. On the other hand, it is possible to show that this manifold can be obtained by a sequence of surgeries on knots with d's less than n, see [16] for details. □

Now let us formulate the connection between the renormalized Euler characteristic and Turaev torsion. For rational homology three-sphere M and a Spin^c structure $\mathfrak{t}$ on it define

$$\widehat{\tau}(M,\mathfrak{t}) = -\tau(M,\mathfrak{t}) + \lambda(M).$$

Theorem 3.4. *For any rational homology three-sphere M and a* Spin^c *structure $\mathfrak{t}$ on it we have*

$$\widehat{\chi}(M,\mathfrak{t}) = \widehat{\tau}(M,\mathfrak{t}) = SW(M,\mathfrak{t}).$$

Proof. The proof follows using the framework of [10]. According to it, there are several conditions on $\widehat{\chi}$ and χ^{trunc} that guarantee the sought equality. We list them as follows:

- The surgery formula of Theorem 3.1 is satisfied. Note that we have a negative sign in front of the second term, but it can be made positive by switching from χ^{trunc} to $-\chi^{\mathrm{trunc}}$.
- For any three-manifold M with $b_1(M) = 1$ and a Spin^c structure $\mathfrak{t}$ on it

$$-\chi^{\mathrm{trunc}}(M,\mathfrak{t}) = \tau(M,\mathfrak{t}).$$

- For any rational homology sphere M,

$$\sum_{\mathfrak{t}\in \mathrm{Spin}^c(M)} \widehat{\chi}(M,\mathfrak{t}) = \left|H_1(M;\mathbb{Z})\right|\lambda(M).$$

- For any integral homology sphere M

$$\widehat{\chi}(M,\mathfrak{t}_0) = \widehat{\tau}(M,\mathfrak{t}_0),$$

  where $\mathfrak{t}_0$ is the unique Spin^c structure on M.
- When M is a Lens space

$$\widehat{\chi}(M,\mathfrak{t}) = \widehat{\tau}(M,\mathfrak{t}),$$

  for any Spin^c structure $\mathfrak{t}$ on M.
- If M_1 and M_2 satisfy $\widehat{\chi} = \widehat{\tau}$ then so does $M_1 \# M_2$.

The first three facts have already been mentioned, while the fourth item is Theorem 5.1 of [14], the fifth condition is satisfied by [9]. The last statement follows from additivity of d, see Theorem 4.3 of [14] and from a Kunneth type formula, see Corollary 6.3 of [12]. The theorem follows. □

4. Proof of the surgery formula

Let θ^c denote the three-dimensional Spin^c homology bordism group, defined as the set of equivalence classes of pairs $(M, \mathfrak{t})$ where M is a rational homology three-sphere, and $\mathfrak{t}$ is a Spin^c structure over M, with the equivalence given as follows. Ne say $(M_1, \mathfrak{t}_1) \sim (M_2, \mathfrak{t}_2)$ if there is a (connected, oriented, smooth) cobordism N from M_1 to M_2 with $H_i(N, \mathbb{Q}) = 0$ for $i = 1$ and 2, which can be endowed with a Spin^c structure $\mathfrak{s}$ whose restrictions to M_1 and M_2 are $\mathfrak{t}_1$ and $\mathfrak{t}_2$ respectively. The connected sum operation makes this set an Abelian group (whose unit is S^3 with its unique Spin^c structure). The invariant $d(M, \mathfrak{t})$ gives a group homomorphism

$$d\colon \theta^c \longrightarrow \mathbb{Q}.$$

It is proved in [14] that d is a lift of the classical homomorphism

$$\rho\colon \theta^c \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

(see [1]) defined as follows. Let N be any four-manifold equipped with a Spin^c structure $\mathfrak{s}$ with $\partial N \cong M$ and $\mathfrak{s}|\partial N \cong \mathfrak{t}$ then

$$\rho(M, \mathfrak{t}) \equiv \frac{c_1(\mathfrak{s})^2 - \operatorname{sgn}(N)}{4} \pmod{2\mathbb{Z}}$$

where $\operatorname{sgn}(N)$ denotes the signature of the intersection form of N.

Going back to our surgery notation, let W be the standard cobordism between Y and $Y_{p/q}$ obtained by 2-handle additions. Let $\rho'(Y, \mathfrak{t}) \equiv \rho(Y, \mathfrak{t}) \pmod{2\mathbb{Z}}$ such that $\rho'(Y, \mathfrak{t}) \in [0, 2)$. For the manifold $Y_{p/q}$ and a Spin^c structure $\mathfrak{t}$ on it consider any $\mathfrak{s}$ on W with $\mathfrak{s}|Y_{p/q} = \mathfrak{t}$. We define $\rho'(Y_{p/q}, \mathfrak{t}) = \rho'(Y, \mathfrak{s}|Y)$.

For any constant k define

$$HF^+_{\preceq k}(Y_{p/q}, [\mathfrak{a}]) = \bigoplus_{\mathfrak{t} \in \operatorname{Spin}^c(Y_{p/q}; \mathfrak{a})} \bigoplus_{\{d \in \mathbb{Q} \,\big|\, d \leq k + \rho'(Y_{p/q}, \mathfrak{t})\}} HF^+_d(Y_{p/q}, \mathfrak{t}).$$

Y_0 is not a rational homology sphere, if $\mathfrak{t}$ is torsion Spin^c structure on Y_0 one can still define $\rho'(Y_0, \mathfrak{t})$ similarly to above. It is useful to note that equivalence

$$d(Y_0, \mathfrak{t}) \equiv 1 + \frac{c_1(\mathfrak{s})^2 + \operatorname{sgn}(W)}{4} + \rho'(Y_0, \mathfrak{t}) \pmod{2\mathbb{Z}}$$

holds for any $\mathfrak{s} \in \operatorname{Spin}^c(W)$ satisfying $\mathfrak{s}|Y_0 = \mathfrak{t}$, this follows from the grading shift formula for maps induced by cobordisms. One should look at both absolute $\mathbb{Q}$ and $\mathbb{Z}/2\mathbb{Z}$ grading shifts.

Let $\mathfrak{T}$ be the subset of torsion Spin^c structures of $\operatorname{Spin}^c(Y_0)$. Now set

$$HF^+_{\preceq k}(Y_0, [\mathfrak{a}]) = \bigoplus_{\mathfrak{t} \in \operatorname{Spin}^c(Y_0; \mathfrak{a}) \setminus \mathfrak{T}} HF^+(Y, \mathfrak{t}) \oplus \bigoplus_{\mathfrak{t} \in \operatorname{Spin}^c(Y_0; \mathfrak{a}) \cap \mathfrak{T}} \bigoplus_{\{d \in \mathbb{Q} \,\big|\, d \leq k + \rho'(Y_0, \mathfrak{t}\}} HF^+_d(Y_0, \mathfrak{t}).$$

Lemma 4.1. *For integers p,q,d,y with p and q relatively prime, $d>0$ and $0\leq y<d$, there is quantity $k(p,q,d,y)$ with the following property. Let everything be as in Theorem 3.1, then*

$$\chi(HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}])-N\cdot|\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})| =$$
$$= \sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})}\widehat{\chi}(Y_{p/q},\mathfrak{b})+p\sum_{\mathfrak{c}\in\mathrm{Spin}^c(Y;\mathfrak{a})}\frac{\rho'(Y,\mathfrak{c})}{2}+k(p,q,d,y).$$

Proof. (cf. lemma 4.8 in [15].) For sufficiently large N, $HF^+_{\mathrm{red}}(Y_{p/q},[\mathfrak{a}])$ is contained in $HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}])$. Over $\mathbb{Z}$, we have a splitting

$$HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}])\cong HF^+_{\mathrm{red}}(Y_{p/q},[\mathfrak{a}])\oplus(\mathrm{Im}\pi\cap HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}])).$$

But it follows readily from the structure of $HF^\infty(Y_{p/q})$ (c.f. Equation (1)) that

$$\chi(\mathrm{Im}\pi\cap HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}]))=$$
$$=\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})}\#\{[d(Y_{p/q},\mathfrak{b}),2N+\rho'(Y_{p/q},\mathfrak{b})]\cap(d(Y_{p/q},\mathfrak{b})+2\mathbb{Z})\subset\mathbb{Q}\}$$
$$=\sum_{\{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})\}}\left(N+1-\left\lceil\frac{d(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b})}{2}\right\rceil\right),$$

where here $\lceil x\rceil$ denotes the smallest integer greater than or equal to x. Thus we get that

$$\chi(HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}]))-N\cdot|\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})|=$$
$$=\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})}\left(HF^+_{\mathrm{red}}(Y_{p/q},\mathfrak{b})-\left\lceil\frac{d(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b})}{2}\right\rceil+1\right)$$
$$=\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})}\left(HF^+_{\mathrm{red}}(Y_{p/q},\mathfrak{b})-\frac{d(Y_{p/q},\mathfrak{b})}{2}\right)+$$
$$+\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})}\left(\frac{d(Y_{p/q},\mathfrak{b})}{2}-\left\lceil\frac{d(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b})}{2}\right\rceil+1\right).$$

To complete the proof we have to show that the difference

$$k=\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})}\left(\frac{d(Y_{p/q},\mathfrak{b})}{2}-\left\lceil\frac{d(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b})}{2}\right\rceil+1\right)-p\cdot\sum_{\mathfrak{c}\in\mathrm{Spin}^c(Y;\mathfrak{a})}\frac{\rho'(Y,\mathfrak{c})}{2}$$

depends only on p,q,d,y. Clearly

$$k=\sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_{p/q};\mathfrak{a})}\left(\frac{d(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b})}{2}-\left\lceil\frac{d(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b})}{2}\right\rceil+1\right).$$

This in turn depends only on $d(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b}) \pmod{2\mathbb{Z}} = \rho(Y_{p/q},\mathfrak{b})-\rho'(Y_{p/q},\mathfrak{b}) \pmod{2\mathbb{Z}}$ which is completely determined by the collection of all $c_1(\mathfrak{s})^2 \pmod{8\mathbb{Z}}$ with $\mathfrak{s}\in\mathrm{Spin}^c(W)$ satisfying $\mathfrak{s}|Y\in\mathrm{Spin}^c(Y;\mathfrak{a})$. This follows from the definitions and the fact that ρ is a homomorphism. Hence, the proof is concluded by the following lemma.

Lemma 4.2. *Let W be the standard cobordism between Y and $Y_{p/q}$. The collection with repetitions of all $c_1(\mathfrak{s})^2$ satisfying $\mathfrak{s}\in\mathrm{Spin}^c(W)$ and $\mathfrak{s}|Y\in\mathrm{Spin}^c(Y;\mathfrak{a})$ is completely determined by the values of p,q,d and y.* □

□

Lemma 4.3. *For integers d,y with $d>0$ and $0\le y<d$, there is quantity $r(d,y)$ with the following property. Let everything be as in Theorem 3.1, then*

$$\chi(HF^+_{\preceq 2N}(Y_0,[\mathfrak{a}]) = \sum_{\mathfrak{b}\in\mathrm{Spin}^c(Y_0;\mathfrak{a})} \chi^{\mathrm{trunc}}(Y_0,\mathfrak{b}) + r(d,y).$$

Proof. The idea of the proof is the same with the previous one. We do not have any terms involving N because of the different structure of HF^∞ for manifolds with $b_1=1$. □

Lemma 4.4. *For integers p,q,d,y with $p\neq 0$, p and q relatively prime, $d>0$ and $0\le y<d$, there is quantity $c(p,q,d,y)$ with the following property. Let everything be as in Theorem 3.1, then*

$$\chi(HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}])) = p\cdot\chi(HF^+_{\preceq 2N}(Y,[\mathfrak{a}])) - q\cdot\chi(HF^+_{\preceq 2N}(Y_0,[\mathfrak{a}])) + c(p,q,d,y), \quad (3)$$

provided that N is sufficiently large.

Proof. The proof is a generalization of the argument of lemma 4.9 in [15]. Let us use induction on $p+q$. The base of induction is the case when $p+q=1,2$, which reduces to $(p,q)=(1,0)$ or $(1,1)$. The lemma clearly holds for the first combination; we will discuss the second case in the end of the proof.

For a pair (p,q) of relatively prime, non-negative integers with $p+q>2$, one can select two pairs of non-negative, relatively prime integers (p_0,q_0) and (p_2,q_2), with $p_0,p_2\neq 0$ satisfying

$$p_0\cdot q - p\cdot q_0 = -1 \quad (4)$$
$$(p,q) = (p_0,q_0)+(p_2,q_2) \quad (5)$$

Consider the manifolds Y_{p_0/q_0}, $Y_{p/q}$ and Y_{p_2/q_2}. There are standard 2-handle cobordisms between these manifolds. Let W_0 denote the cobordism between Y_{p_0/q_0} and $Y_{p/q}$, W_1 the cobordism between $Y_{p/q}$ and Y_{p_2/q_2}, W_2 between Y_{p_2/q_2} and Y_{p_0/q_0}. We can write down the following long exact sequence

$$\ldots \longrightarrow HF^+(Y_{p_0/q_0},[\mathfrak{a}]) \xrightarrow{f_0} HF^+(Y_{p/q},[\mathfrak{a}]) \xrightarrow{f_1} HF^+(Y_{p_2/q_2},[\mathfrak{a}]) \xrightarrow{f_2} \ldots,$$

where the maps are induced by the corresponding cobordisms. Note that W_0 and W_1 are both negative definite, but W_2 is not.

By inductive hypothesis the lemma holds for (p_0, q_0) and (p_2, q_2). When N is sufficiently large, the image of the restriction g_0 of f_0 to $HF^+_{\preceq 2N}(Y_{p_0/q_0}, [\mathfrak{a}])$ is contained in $HF^+_{\preceq 2N+\frac{1}{4}}(Y_{p/q}, [\mathfrak{a}])$, the restriction g_1 of f_1 to $HF^+_{\preceq 2N+\frac{1}{4}}(Y_{p/q}, [\mathfrak{a}])$ is contained in $HF^+_{\preceq 2N+\frac{1}{2}}(Y_{p_2/q_2}, [\mathfrak{a}])$, and finally, the restriction g_2 of f_2 to $HF^+_{\preceq 2N+\frac{1}{2}}(Y_{p_2/q_2}, [\mathfrak{a}])$ is contained in $HF^+_{\preceq 2N}(Y_{p_0/q_0}, [\mathfrak{a}])$. This follows at once from the definition of ρ' which appears in the expression for $HF^+_{\preceq k}$, and the grading shift formula: we have that $\chi(W_i) = 1$ and $\sigma(W_i) = -1$ for $i = 0, 1$; while the cobordism W_2 induces the trivial map on HF^∞ since $b_2^+(W_2) = 1$.

Choosing N as above, consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "g_2"] & HF^+_{\preceq 2N}(Y_{p_0/q_0}, [\mathfrak{a}]) \arrow[r, "g_0"] \arrow[d] & HF^+_{\preceq 2N+\frac{1}{4}}(Y_{p/q}, [\mathfrak{a}]) \arrow[r, "g_1"] \arrow[d] & HF^+_{\preceq 2N+\frac{1}{2}}(Y_{p_2/q_2}, [\mathfrak{a}]) \arrow[r, "g_2"] \arrow[d] & \cdots \\
\cdots \arrow[r, "f_2"] & HF^+(Y_{p_0/q_0}, [\mathfrak{a}]) \arrow[r, "f_0"] \arrow[d] & HF^+(Y_{p/q}, [\mathfrak{a}]) \arrow[r, "f_1"] \arrow[d] & HF^+(Y_{p_2/q_2}, [\mathfrak{a}]) \arrow[r, "f_2"] \arrow[d] & \cdots \\
\cdots \arrow[r, "h_2"] & HF^+_{\succ 2N}(Y_{p_0/q_0}, [\mathfrak{a}]) \arrow[r, "h_0"] \arrow[d] & HF^+_{\succ 2N+\frac{1}{4}}(Y_{p/q}, [\mathfrak{a}]) \arrow[r, "h_1"] \arrow[d] & HF^+_{\succ 2N+\frac{1}{2}}(Y_{p_2/q_2}, [\mathfrak{a}]) \arrow[r, "h_2"] \arrow[d] & \cdots, \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns are exact. Note that the first and the third rows are not necessarily exact, while the middle one is exact. Let us think of these three rows as chain complexes. We denote these three rows by $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$. Since $\mathcal{R}_2$ is exact, it follows that $H_*(\mathcal{R}_1) \cong H_*(\mathcal{R}_3)$.

Now let us show that $H_*(\mathcal{R}_3)$ is determined by p, q, d and y for N sufficiently large. This is established using the structure of maps on HF^∞, lemma 4.2 and the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "h_2^\infty"] & HF^\infty_{\succ 2N}(Y_{p_0/q_0}, [\mathfrak{a}]) \arrow[r, "h_0^\infty"] \arrow[d, "\cong"'] & HF^\infty_{\succ 2N+\frac{1}{4}}(Y_{p/q}, [\mathfrak{a}]) \arrow[r, "h_1^\infty"] \arrow[d, "\cong"'] & HF^\infty_{\succ 2N+\frac{1}{2}}(Y_{p_2/q_2}, [\mathfrak{a}]) \arrow[r, "h_2^\infty"] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r, "h_2"] & HF^+_{\succ 2N+}(Y_{p_0/q_0}, [\mathfrak{a}]) \arrow[r, "h_0"] & HF^+_{\succ 2N+\frac{1}{4}}(Y_{p/q}, [\mathfrak{a}]) \arrow[r, "h_1"] & HF^+_{\succ 2N+\frac{1}{2}}(Y_{p_2/q_2}, [\mathfrak{a}]) \arrow[r, "h_2"] & \cdots,
\end{tikzcd}$$

where here h_0 is the sum over all $\mathfrak{s} \in \mathrm{Spin}^c(W_0)$ of the projections of the induced maps on HF^∞; e.g. letting

$$\Pi_{\succ 2N+\frac{1}{2}} \colon HF^\infty(Y_{p/q}, [\mathfrak{a}]) \longrightarrow HF^\infty_{\succ 2N+\frac{1}{2}}(Y_{p/q}, [\mathfrak{a}])$$

denote the projection, we let h_0^∞ be the restriction to $HF_{\succ 2N}(Y_{p_0/q_0},[\mathfrak{a}])$ of

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(W_0)} \Pi_{\succ 2N+\frac{1}{2}} \circ F^\infty_{W_0,\mathfrak{s}}.$$

The maps h_i^∞ are defined similarly. Note that $h_2^\infty = 0$, since the map induced by W_2 has $b_2^+(W_2) = 1$.

So far we have established that for all sufficiently large N,

$$\chi(H_*(\mathcal{R}_1)) = \chi(HF^+_{\preceq 2N}(Y_{p_0/q_0},[\mathfrak{a}])) - \chi(HF^+_{\preceq 2N+\frac{1}{4}}(Y_{p/q},[\mathfrak{a}])) + \chi(HF^+_{\preceq 2N+\frac{1}{2}}(Y_{p_2/q_2},[\mathfrak{a}]))$$

is completely determined by p, q, d, y. It is also clear that for sufficiently large N,

$$\chi(HF^+_{\preceq 2N+\frac{1}{4}}(Y_{p/q},[\mathfrak{a}])) = \chi(HF^+_{\preceq 2N}(Y_{p/q},[\mathfrak{a}])) + c_3$$

and

$$\chi(HF^+_{\preceq 2N+\frac{1}{2}}(Y_{p_2/q_2},[\mathfrak{a}])) = \chi(HF^+_{\preceq 2N}(Y_{p_2/q_2},[\mathfrak{a}])) + c_4,$$

with constants c_3 and c_4 again depending only on p_0, q_0, d, y and p_2, q_2, d, y respectively, lemma 4.2. Combining all the constants, we establish the inductive step in the case where p_0 is non-zero.

When $(p, q) = (1, 1)$, the above argument works with slight modification. In this case, we consider the manifolds Y, Y_0, Y_1. The dimension shifts work differently: $\sigma(W_0) = \sigma(W_1) = 0$ and hence, we compare $HF^+_{\preceq 2N}(Y,[\mathfrak{a}])$, $HF^+_{\preceq 2N+\frac{1}{2}}(Y_0,[\mathfrak{a}])$, and $HF^+_{\preceq 2N+1}(Y_1,[\mathfrak{a}])$. To see that the map f_2 induced by W_2 carries $HF^+_{\preceq 2N+1}(Y_1,[\mathfrak{a}])$ into $HF^+_{\preceq 2N}(Y,[\mathfrak{a}])$ for sufficiently large N, remember that the kernel of the map f_0 induced by W_0 is finitely generated. Some parities change under these maps, so the Euler characteristic is given as follows

$$\chi(H_*(\mathcal{R}_1)) = \chi(HF^+_{\preceq 2N}(Y,[\mathfrak{a}])) - \chi(HF^+_{\preceq 2N+\frac{1}{4}}(Y_0,[\mathfrak{a}])) - \chi(HF^+_{\preceq 2N+\frac{1}{2}}(Y_1,[\mathfrak{a}])),$$

compare with Proposition 5.3 in [14].

□

Proof of Theorem 5.3.1 When p and q are non-negative, this is a combination of Lemmas 4.1 and 4.4. The remaining case can be proved by running the induction from Lemma 4.4 to show that it still holds in the case where $p > 0$ and $q \leq 0$. □

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The Program in Applied and Computational Mathematics, Princeton University, New Jersey 08540, USA

E-mail address: `rustamov@princeton.edu`